CRITICAL LEVELS AND JACOBI FIELDS IN A COMPLEX OF CYCLES

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ABSTRACT. Suppose S is a closed, oriented, connected surface of genus at least two. In this paper a complex of cycles is shown to have have the property that the space of tight geodesic paths connecting any two vertices has uniformly bounded dimension. This complex of cycles, $\mathcal{C}(S, \alpha)$, is a variant of complexes also studied by Bestvina-Bux-Margalit and Hatcher-Margalit, and has been used to prove theorems about the Torelli group and surfaces in $S \times I$, (Bestvina-Bux-Margalit, Hatcher-Margalit, Irmer). Conjugate points in the space of tight geodesics arise from critical levels of a function related to distance, and are used to generalise Masur-Minsky's notion of subsurface projection. When studying families of tight geodesic paths, it is seen that paths can become "stuck" in subspaces. This observation is used to define cusps and horospheres in $\mathcal{C}(S, \alpha)$, which are isometrically embedded subcomplexes closed under the action of corresponding stabiliser subgroups of the Torelli group.

1. INTRODUCTION

Let α be a primitive, nontrivial element of $H_1(S, \mathbb{Z})$. The *complex of cycles*, $\mathcal{C}(S, \alpha)$, is a complex whose vertex set is the set of all homotopy classes of oriented multicurves in S in the homology class α, where the multicurves are not allowed to have null homologous submulticurves. Two vertices v_1 and v_2 are joined by an edge if the corresponding multicurves are isotopic to the oriented boundary of an embedded subsurface of S with the subsurface orientation. Higher dimensional simplices can also be defined as in [2] for example, however they will not be needed in this paper.

The *distance*, $d_{\mathcal{C}}(v_1, v_2)$, between v_1 and v_2 in $\mathcal{C}(S, \alpha)$ is defined to be the distance in the path metric of the one-skeleton, where all edges have length one.

The *Torelli group* $\mathcal{T}$ of S is the subgroup of the mapping class group of S that acts trivially on $H_1(S, \mathbb{Z})$. The complex $\mathcal{C}(S, \alpha)$ is a member of a family of complexes that generalise Harvey's Complex of Curves,

$\mathcal{C}(S)$, to study $\mathcal{T}$. For example, in [2] it was used to calculate cohomological properties of $\mathcal{T}$, and in [1] a related complex was used to reprove a result of Birman-Powell about the generating set of the Torelli group of a surface with genus at least three.

Curve complexes are in general locally infinite, so there can be infinitely many geodesic arcs connecting two verticies. In order to be able to prove theorems in a locally infinite complex, the concept of tightness was introduced in [9] and modified in [4]. In [9] and [4] it was shown that there are only finitely many tight geodesics connecting any two verticies v_1 and v_2 in $\mathcal{C}(S)$. Subsection 1.1 gives a definition of tightness for $\mathcal{C}(S, \alpha)$, which is equivalent to the definition of [4] when restricted to $\mathcal{C}(S)$.

In $\mathcal{C}(S, \alpha)$, it follows from the main theorem of this paper that there are finitely many tight geodesics connecting any two vertices; however, unlike in $\mathcal{C}(S)$, this is not a consequence of hyperbolicity, and geodesics do not fellow travel in this complex. There is extra structure on $\mathcal{C}(S, \alpha)$, which makes it possible to define a finite set of optional surgeries that can be performed on the multicurves representing the vertices of a geodesic path to obtain another geodesic path connecting the same endpoints. Geometrically, these optional surgeries behave like a discrete analogue of a finite space of Jacobi fields along a tight geodesic path.

Theorem 1. *Given any two vertices, v_1 and v_2 in $\mathcal{C}(S, \alpha)$, the space of tight geodesics connecting v_1 and v_2 has dimension bounded from above by $-3\chi(S)$.*

In [7], given two homologous multicurves m_1 and m_2, the overlap function $f(m_1, m_2) : S \diagdown (m_1, m_2) \to \mathbb{Z}$, was defined to study the geometry of a complex closely related to $\mathcal{C}(S, \alpha)$. In the absence of hyperbolicity, [7], the most natural tools in $\mathcal{C}(S, \alpha)$ seem to come from discrete Morse theory. It therefore stands to reason that critical levels will be important in this context. There is a notion of critical level of the overlap function. The existence of critical levels are a necessary condition for there to be choices involved in the construction of tight geodesic paths connecting a pair of vertices. The finite number of critical levels, and hence the finite dimensionality of the space of tight geodesics connecting any two vertices in $\mathcal{C}(S, \alpha)$, follows simply from an Euler characteristic argument on the surface S.

Sublevel Projection. The Masur-Minsky notion of subsurface projection is not directly applicable to many problems arising from studying $\mathcal{C}(S, \alpha)$. There is not in general a sense in which two multicurves are homologous inside a given subsurface, and distances in $\mathcal{C}(S, \alpha)$ also depend on the number and relative orientations of arcs in subsurfaces. Questions relating to the way the Torelli group restricts to subsurfaces have already been shown to be central to understanding generating sets of the Torelli group, [10]. In section 3 a notion analogous to subsurface projection from [9] is defined using conjugate points arising from the critical levels. A distance formula analogous to that in [9] follows from the finite number of critical levels and distance calculations in [7].

Some of the available choices involved in constructing tight geodesic paths make it possible for a subpath to remain within in a subspace of S. This is used in section 5 to define cusps and horoballs in $\mathcal{C}(S, \alpha)$. A horoball is an isometrically embedded subcomplex of $\mathcal{C}(S, \alpha)$ closed under the action of a corresponding stabiliser subgroup of the Torelli group. Outside of the cusps, the orbit of any vertex under the action of the Torelli group comes within a fixed diameter of any other vertex.

1.1. **Notations and Definitions.** A *curve c* in S is a piecewise smooth, injective map of S^1 into S that is not null homotopic. A *multicurve* is a union of pairwise disjoint curves on S. When convenient, a multicurve is confused with its image in S. Whenever this does not lead to confusion, the same symbol will also be used for a vertex in $\mathcal{C}(S, \alpha)$ and a representative of the corresponding homotopy class of multicurves on S.

Since S is supposed to be oriented, assume all subsurfaces have the subsurface orientation. If a null homologous multicurve n bounds an embedded subsurface of S, the union of the components of $S \smallsetminus n$ whose boundary orientation coincides with the orientation of n will be called the *subsurface of S bounded by n*.

A path in $\mathcal{C}(S, \alpha)$ will be denoted by listing a set of multicurves representing the vertices of the path. For example, when $\{\gamma_1, \ldots, \gamma_n\}$ are multicurves in the integral homology class α, and $i(\gamma_i, \gamma_{i+1}) = 0 \ \forall i$, $\{\gamma_1, \ldots, \gamma_n\}$ denotes a path connecting γ_1 and γ_n in $\mathcal{C}(S, \alpha)$.

Two multicurves m_1 and m_2 in general position are said to *fill S* if their complement in S is a union of discs.

In [4], a path in $\mathcal{C}(S)$ was called *tight* at the index $\{i \neq 0, n\}$ if, for every curve c in γ_i, every curve on the surface S that crosses c also crosses some element of $\gamma_{i-1} \cup \gamma_{i+1}$. Informally, this definition ensures that γ_i is contained within or on the boundary of the connected subspace of S filled by $\gamma_{i-1} \cup \gamma_{i+1}$. Recall that any two multicurves representing vertices in $\mathcal{C}(S)$ separated by a distance at least three automatically fill S. For a geodesic path $\{\gamma_1, \ldots, \gamma_n\}$ in $\mathcal{C}(S)$, it therefore automatically follows from the definition that γ_i is contained within or on the boundary of the connected subspace of S filled by $\gamma_j \cup \gamma_k$, for all $j < i$ and $k > i$.

However, for $\mathcal{C}(S, \alpha)$, vertices separated by an arbitrarily large distance do not necessarily fill S. A simple example illustrating this is example 3; the distance between the two curves γ_1 and γ_n was shown in [7] to be equal to half the intersection number plus one. For this reason, a path $\{\gamma_1, \ldots, \gamma_n\}$ in $\mathcal{HS}(S, [\gamma_1])$ is defined to be *tight* if, for every curve c in γ_i, every curve on the surface S that crosses c also crosses some element of $\gamma_j \cup \gamma_k$, for all $j < i$ and $k > i$. This definition then rules out the possiblity that a subpath of a tight geodesic enters a subsurface of S that the two endpoints of the path do not enter.

2. Basic geodesic construction

An algorithm for constructing geodesic paths in a closely related complex was given in [7]. Since some of these ideas will be used in this paper for studying families of geodesics, the general approach is outlined in this section.

The *overlap function*, also denoted by the symbol f, of a null homologous union of curves, n, is a locally constant integer valued function defined on $S \smallsetminus n$ with minimum value zero. For any two points x and y in $S \smallsetminus n$, $f(x) - f(y)$ is the algebraic intersection number of n with an oriented arc with starting point y and endpoint x. An important special case is the overlap function of the difference of two homologous multicurves, $m_2 - m_1$.

The overlap function is not dependent on the choice of oriented arc, because the algebraic intersection number of any closed loop with n is zero. It does however depend on the choice of representatives of the homotopy classes of curves. It will be assumed that the representatives of the homotopy classes are chosen so that the maximum, M, of

the overlap function is as small as possible. When n does not contain homotopic curves, it is sufficient to assume that the curves in n are in general and minimal position. For two homologous multicurves m_1 and m_2, the quantity M will be called the *homological distance, $\delta(m_1, m_2)$,* between m_1 and m_2.

Corollary 2 (Corollary of theorem 4 of [7])**.** *Let m_1 and m_2 be two multicurves corresponding to vertices of $\mathcal{C}(S, \alpha)$. Then $d_{\mathcal{C}}(m_1, m_2) = \delta(m_1, m_2)$.*

The proof of this corollary will be left until after the general construction has been outlined.

In order to construct paths in $\mathcal{HC}(S, \alpha)$, it is necessary to use some properties of the level sets, in particular the local extrema, of the overlap function. These will be used to define the surgeries used in the path construction algorithm.

Given an oriented multicurve a with a regular neighbourhood $\mathcal{N}(a)$ and an orientation on S, the left and right component of $\mathcal{N} \setminus a$ can be defined. If b is an oriented multicurve that intersects a transversely at a point p, it therefore makes sense to say that b *crosses over a from left to right* (or *right to left*) at p. Similarly, if b is an oriented arc with an endpoint on a, a notion in which b *leaves or approaches a from the left or right* can be defined.

If an arc of $a \cap (S \setminus b)$ leaves and approaches b from the right, then this arc is *to the right of b* and vice versa.

Whenever m_1 and m_2 are homologous multicurves, the overlap function of m_1 and m_2 is bounded and has a maximum. Call the subsurface of S on which the overlap function takes on its maximum $S_{max}(m_1, m_2)$. The boundary of $S_{max}(m_1, m_2)$ consists of arcs of m_1 and m_2 such that $S_{max}(m_1, m_2)$ is to the right of any arc of m_1 on its boundary and to the left of any arc of m_2 on its boundary. In other words, the boundary of $S_{max}(m_1, m_2)$ is a null homologous multicurve made up of arcs of m_1 to the left of m_2 and arcs of m_2 to the right of m_1.

Similarly, the subsurface of S, $S_{min}(m_1, m_2)$, on which $f = 0$, is disjoint from $S_{max}(m_1, m_2)$ and is on the left of any arc of m_1 on its boundary and to the right of any arc of m_2 on its boundary.
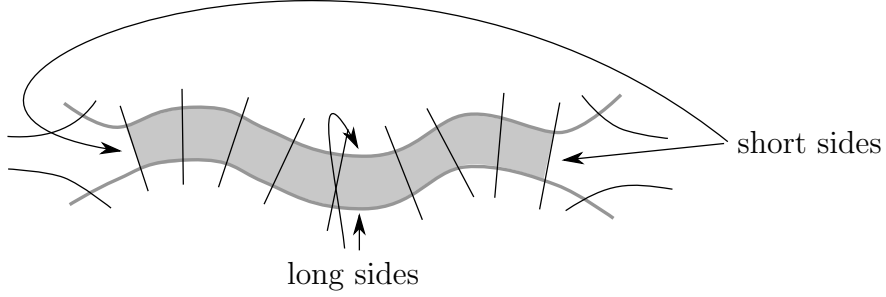
FIGURE 1. The rectangle representing a homotopy class of arcs.

Horizontal and vertical arcs Given two multicurves a and b on an oriented surface S, a *horizontal arc* of a is a component of $a \cap (S \setminus b)$ that leaves and approaches b from the same side. A *vertical arc* of $a \cap (S \setminus b)$ leaves and approaches b from opposite sides. An "innermost" arc in [6] is an example of a horizontal arc.

Informally, a horizontal arc is an arc alongside a component of $S \setminus (m_1 \cup m_2)$ where the overlap function has a local extremum in some direction, whereas a vertical arc denotes a direction in which the overlap function is increasing or decreasing.

Suppose a and b are multicurves in S in general position. Two arcs a_1 and a_2 of $a \cap (S \setminus b)$ will be called homotopic if the closure of a_1, $\bar{a}_1$, can be homotoped onto the closure of a_2, $\bar{a}_2$, by a homotopy that keeps the endpoints of the arcs on b. Since an arc of a is defined to be a connected component of $a \cap (S \setminus b)$, a homotopy is also not allowed to move any interior point of the arc over b.

The arcs on $\partial S_{max}(m_1, m_2)$ and $\partial S_{min}(m_1, m_2)$ are all horizontal.

As shown in figure 1, a homotopy class of arcs of $a \cap (S \setminus b)$ can be treated as a rectangle, R_a in S. One pair of opposite sides of the rectangle, the "short" sides, consist of arcs of $a \cap (S \setminus b)$ on the boundary of a *nontrivial* component of $S \setminus (a \cup b)$, in other words, a component of $S \setminus (a \cup b)$ that is not a rectangle or bigon. These nontrivial components are called the *ends* of the homotopy class. If there is only one arc in the homotopy class a, R_a consists of this single arc.

A basic surgery construction. Suppose b is an oriented multicurve, and a_1 and a_2 are two homotopic arcs with endpoints on b. Then
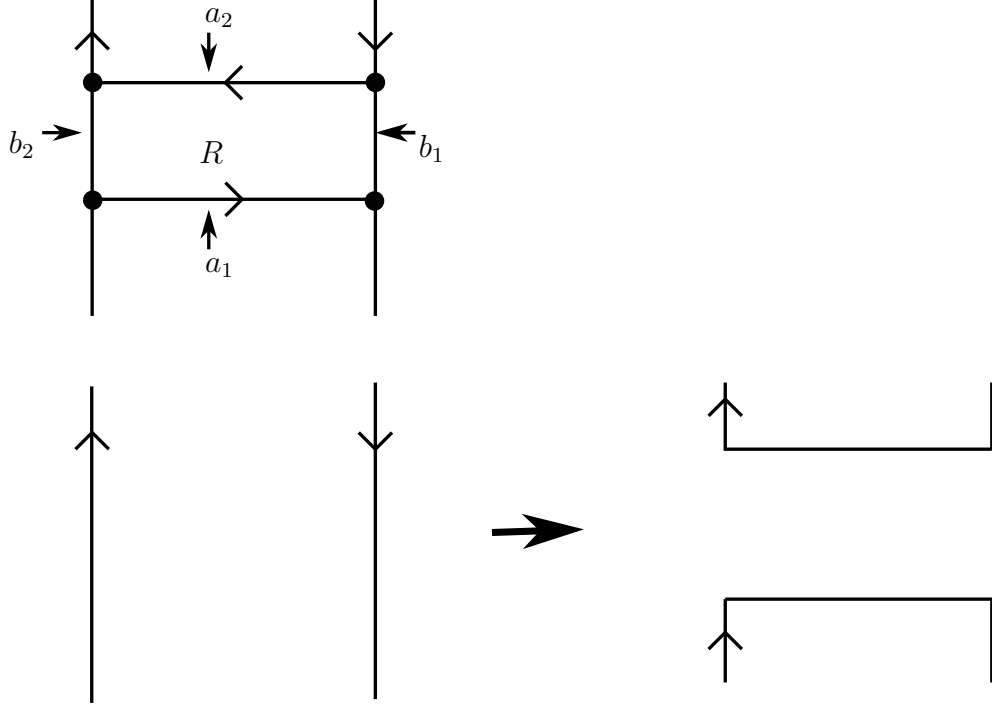
FIGURE 2. Surgering a multicurve along a horizontal arc.

$a_1 \cup a_2 \cup b$ can be thought of as a one dimensional simplicial complex on S. If a_1 is a horizontal arc, the arcs a_1 and a_2 can be oriented in such a way that $a_1 \cup a_2 \cup b_1 \cup b_2$ bound a rectangle R in S, where b_1 and b_2 are chains in $a_1 \cup a_2 \cup b$, as shown in figure 2. *Surgering an oriented multicurve b along a horizontal arc a* is the process in which the oriented chains $-a_1$, $-a_2$, $-b_1$ and $-b_2$ are added to the subcomplex b. Since the chain added is a boundary, the resulting multicurve is homologous to b. It was shown in [6] or [7] that any path in $\mathcal{C}(S, \alpha)$ can be constructed by surgering along horizontal arcs and adding or discarding null homologous multicurves or boundaries of discs.

In order to construct a geodesic path $\gamma_1, \ldots, \gamma_n$ in $\mathcal{C}(S, \alpha)$, given γ_i, theorem 2 implies that γ_{i+1} is constructed by surgering γ_i along horizontal arcs in such a way that the maximum of the overlap function with γ_n is reduced by one. One way of doing this is to perform the *surgery* or *surgeries corresponding to* $S_{max}(\gamma_i, \gamma_n)$ *on* γ_i, as follows:
    The surgeries corresponding to $S_{max}(\gamma_i, \gamma_n)$

(1) The multicurve γ_i is first surgered along the arcs $a_1, a_2, \ldots$, where $a_1, a_2, \ldots$ are the arcs of $\gamma_n \cap (S \smallsetminus \gamma_i)$ on $\partial S_{max}(\gamma_i, \gamma_n)$.

(2) The surgery from (1) gives a multicurve $\gamma'_{i+1} - \partial S_{max}(\gamma_i, \gamma_n)$. Discard the null homologous submulticurve $-\partial S_{max}(\gamma_i, \gamma_n)$ and any contractible submulticurves of γ'_{i+1} to obtain γ_{i+1}.

It is not difficult to check that discarding $-\partial S_{max}(\gamma_i, \gamma_n)$ reduces the maximum of the overlap function with γ_n by one.

The surgeries corresponding to $S_{min}(\gamma_i, \gamma_n)$ are defined analogously.

A *middlepath* connecting the vertices γ_1 and γ_n is a path in which, for all i, γ_{i+1} is obtained from γ_i by performing the surgery corresponding to $S_{max}(\gamma_i, \gamma_n)$. As shown in theorem 9 of [7], this definition is symmetric in γ_1 and γ_n.

By construction, $\partial S_{max}(\gamma_i, \gamma_n)$ does not intersect γ_i, and each connected component of $S_{max}(\gamma_i, \gamma_n)$ intersects an annular neighbourhood of γ_i on the right side of γ_i (i.e. every component of $S_{max}(\gamma_i, \gamma_n)$ is "on the same side" of γ_i). Therefore γ_{i+1}, so constructed, does not intersect γ_i.

A minor complication arises because the surgeries corresponding to $S_{max}(\gamma_i, \gamma_n)$ do not always define an edge in $\mathcal{C}(S, \alpha)$. To prove corollary 2 it is necessary to show that is is alway possible to construct a multicurve γ_{i+1} with the following properties:

(1) $\gamma_{i+1} - \gamma_i$ is the oriented boundary of an embedded subsurface of S with the subsurface orientation.

(2) $\delta(\gamma_{i+1}, \gamma_n) = \delta(\gamma_i, \gamma_n) - 1$, and

(3) γ_{i+1} does not contain any null homologous submulticurves.

Proof of Corollary 2. It might be the case that performing the surgery on γ_i corresponding to $S_{max}(\gamma_i, \gamma_n)$ will give a multicurve with null homologous submulticurves. These null homologous submulticurves can not always simply be discarded without violating the condition that $\gamma_{i+1} - \gamma_i$ is the oriented boundary of an embedded subsurface of S. Note that whenever γ_i and γ_{i+1} are in minimal position, contractible curves do not cause this problem.

Let $Int(n)$ denote the subsurface of S bounded by n. Suppose a null homologous submulticurve n of γ'_{i+1} bounds a subsurface of S that contains some components $S_{max}(\gamma_i, \gamma_n)_1$, $S_{max}(\gamma_i, \gamma_n)_2, \ldots$ of $S_{max}(\gamma_i, \gamma_n)$.
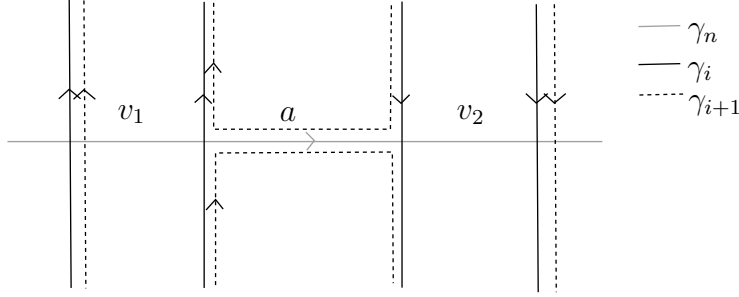
FIGURE 3. Consecutive surgeries.

Whenever $Int(n)$ does not contain any components of $S_{min}(\gamma_i, \gamma_n)$, performing the surgery on γ_i corresponding to $S_{max}(\gamma_i, \gamma_n) \smallsetminus (S_{max}(\gamma_i, \gamma_n)_1, S_{max}(\gamma_i, \gamma_n)_2, \ldots)$ and discarding all null homologous and contractible curves will give a multicurve γ_{i+1} with the desired properties.

Otherwise, let $S_{min}(\gamma_i, \gamma_n)_1, S_{min}(\gamma_i, \gamma_n)_2, \ldots$ be the components of $S_{min}(\gamma_i, \gamma_n)$ in $Int(n)$. Basically, the idea is to perform a surgery corresponding to $Int(n) \smallsetminus (S_{min}(\gamma_i, \gamma_n)_1, S_{min}(\gamma_i, \gamma_n)_2, \ldots)$; it is not hard to check that the relative orientations on all relevant subarcs are oriented in such a way as to make this surgery well defined. Perform the surgeries corresponding to $S_{max}(\gamma_i, \gamma_n) \smallsetminus (S_{max}(\gamma_i, \gamma_n)_1, S_{max}(\gamma_i, \gamma_n)_2, \ldots)$ and $Int(n) \smallsetminus (S_{min}(\gamma_i, \gamma_n)_1, S_{min}(\gamma_i, \gamma_n)_2, \ldots)$ on γ_i to obtain a multicurve γ'_{i+1}. It follows that $\delta(\gamma'_{i+1}, \gamma_n) = \delta(\gamma_i, \gamma_n) - 1$, as required.

Suppose γ'_{i+1} contains the null homologous submulticurves $n_2, n_3, \ldots$. By construction, $Int(n_i)$ is strictly contained in $Int(n)$ for all i. Since γ_i does not have any null homologous submulticurves, and $\gamma'_{i+1} - \gamma_i$ bound a subsurface of S, each n_i could be homotopic to at most one other n_j, for $i \neq j$, where two homotopic null homologous submulticurves of γ'_{i+1} necessarily have opposite orientation. It follows that the construction in the previous paragraph only has to be repeated finitely many times to obtain a γ_{i+1} satisfying all three properties. $\square$

2.1. **Patches.** It would be convenient to have a notion of the subspace of S on which a multicurve has been altered by a surgery, such that it is possible to perform surgeries in different subspaces independently of each other. In order to do this, the patches, P_i are defined. The patch P_i is intended to represent the subspace of S on which the multicurve γ_1 was altered to obtain γ_{i+1}.

Suppose γ_i is surgered along the horizontal arc a of $\gamma_n \cap (S \smallsetminus \gamma_i)$ as shown in diagram 3 to obtain γ_{i+1}. In this example, a horizontal arc $v_1 \circ a \circ v_2$ of $\gamma_n \cap (S \smallsetminus \gamma_{i+1})$ is created, and we would like to think of surgering γ_{i+1} along $v_1 \circ a \circ v_2$ as being the obvious continuation of the first surgery, repeatable until a critical level, as defined in the next section, is reached. Although it is possible to choose representatives of homotopy classes so that this is not the case, there is a sense in which the subsurface of S in which γ_i was altered to obtain γ_{i+1} is contained within the subsurface of S in which γ_{i+1} was altered to obtain γ_{i+2}.

Recall that γ_1 and γ_n are assumed to be in general and minimal position, and suppose γ_{i+1} is obtained from γ_i by surgering along the horizontal arcs $\{a_m\}$ and discarding the null homologous multicurves $\{n_j\}$ and contractible curves $\{c_k\}$. Let $\{d_k\}$ be the disks bounded by the curves $\{c_k\}$, $\{G_l\}$ be the $2n$-gons in $S \smallsetminus (\gamma_i \cup \gamma_n)$, $n > 2$, with the property that $n-1$ sides of each G_l are in the homotopy classes $\{[a_m]\}$, and let b_l be the (necessarily horizontal) arc of $\gamma_n \cap (S \smallsetminus \gamma_i)$ on G_l that is not in any of the homotopy classes $\{[a_m]\}$. Let

$$P_1 := \cup_m R_{a_m} \cup_l (G_l \cup R_{b_l}) \cup_k d_k \cup_j n_j.$$

It is a consequence of tightness that all the arcs a_m are necessarily homotopic to arcs of $\gamma_n \cap (S \smallsetminus \gamma_1)$. By construction, the multicurve γ_2 is homotopic to a multicurve that does not enter P_1, by a homotopy that fixes everything outside of an ϵ-neighbourhood of P_1. The multicurve $\{\gamma_2\}$ is the representative of its homotopy class chosen such that it coincides with γ_1 outside of an ϵ-neighbourhood of P_1. Similarly,

$$P_{i+1} := P_i \cup_i R_{a_i} \cup_l (G_l \cup R_{b_l}) \cup_k d_k \cup_j n_j.$$

The representatives of the homotopy classes of the multicurves $\{\gamma_{i+1}\}$ are chosen such that the curves and arcs of γ_{i+1} coincide with curves and arcs of γ_i outside of an ϵ-neighbourhood of $\cup_i R_{a_i} \cup_l (G_l \cup R_{b_l}) \cup_k d_k \cup_j n_j$.

2.2. **Critical Levels.** This and the next section section are not strictly necessary in the proof of theorem 1 and will only be needed later in section 3, however it is included at this point because it is hoped that the ideas presented here will provide clarification.

Informally, a middle path $\gamma_1, \gamma_2, \ldots, \gamma_n$ is constructed in such a way that the surgeries performed on γ_{i+1} to obtained γ_{i+2} are a continuation of the surgeries performed on γ_i to obtained γ_{i+1}, as discussed in the previous section, unless i is a critical level. For a middle path, when γ_1 and γ_n have large intersection number, for most i, P_{i+1} is obtained
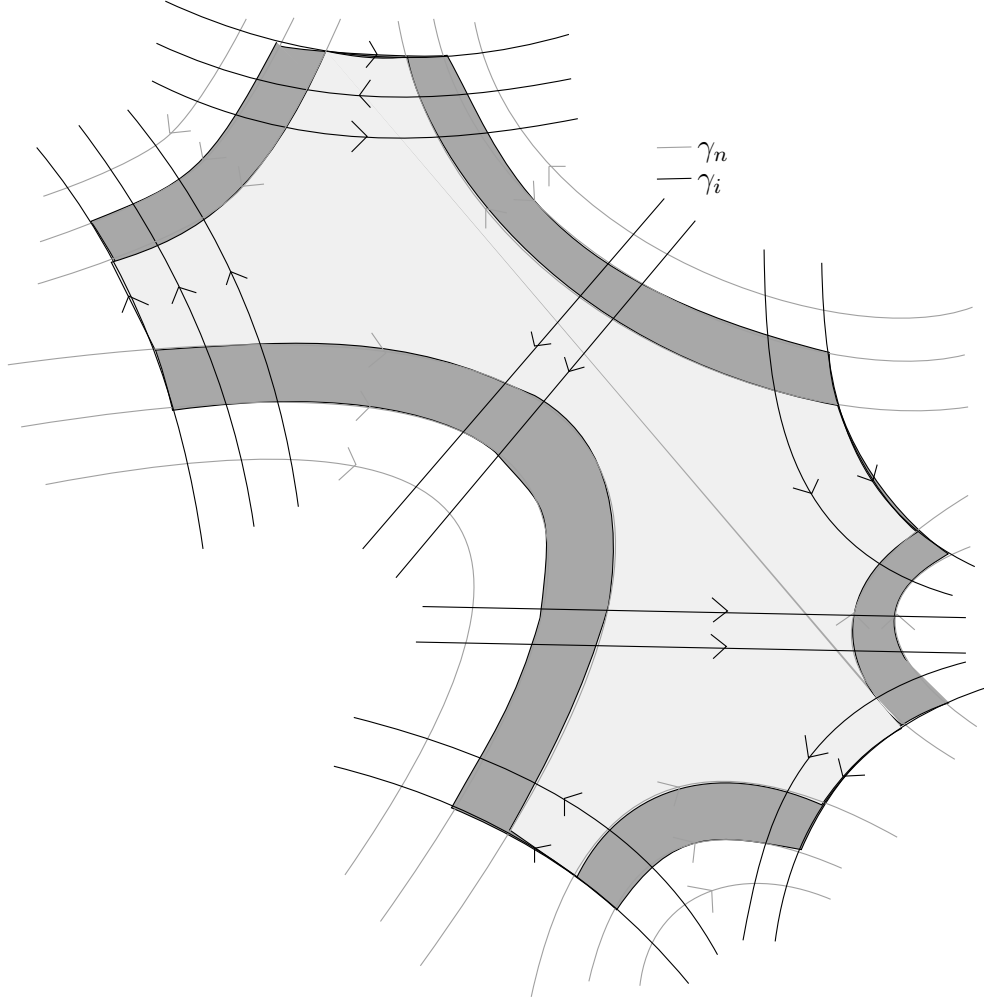
FIGURE 4. The boundary gluing condition.

from P_i by attaching rectangles to the boundary, where each rectangle
is attached along a single boundary arc.

The boundary gluing condition. The patch P_{i+1} satisfies the
boundary gluing condition if every component of P_{i+1} is constructed
from a component of P_i by gluing a rectangle along each arc of γ_i on
∂P_i, as shown in figure 4. Each attached rectangle shares only one
boundary arc with P_i, and is attached along arcs of $\gamma_n \cup (S \smallsetminus \gamma_i)$ to
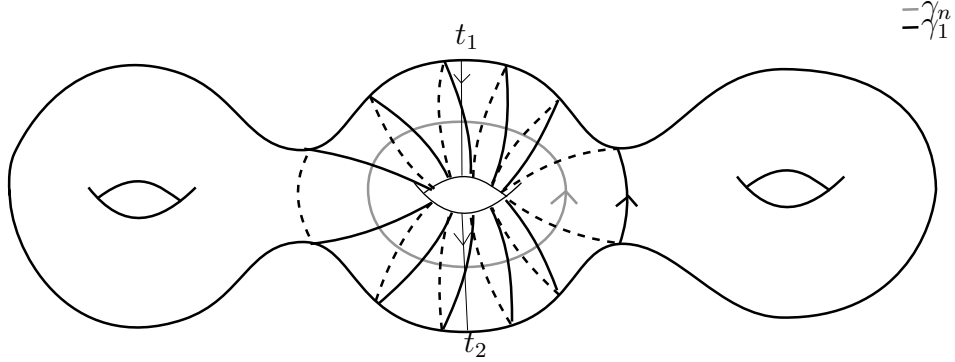rectangles with common vertices on ∂P_i.

FIGURE 5. The endpoints of a path with no critical levels.

For $1 < i < n$, a *noncritical level* of a middle path is a value of i for which P_{i+1} satisfies the boundary gluing condition. A *critical level* is then any $1 < i < n$ that is not noncritical.

When constructing a general tight geodesic, there are often choices involved. A critical level should reflect the fact that some surgery performed previously can not be continued in an obvious way, even when there exist choices for constructing the next multicurve that do. For $1 < i < n$, i is critical level of a tight geodesic path if there exists a component P_i' of P_i, such that there does not exist a possible γ_{i+1} such that P_i' satisfies the boundary gluing condition, where i is the smallest integer for which P_i' is a component of P_i.

A few examples are given to motivate this definition.

Example 3 (A path with no critical levels). *The endpoints γ_1 and γ_n are illustrated in figure 5. A tight geodesic path $\{\gamma_1, \ldots, \gamma_n\}$ can be constructed as follows: γ_{i+1} is obtained by Dehn twisting γ_i around the disjoint curves t_1 and t_2 shown in the figure. It is not hard to show that this is the only tight geodesic path connecting γ_1 and γ_n. There are no critical levels in this example; for each $1 < i < n$, P_{i+1} satisfies the boundary gluing condition.*

Example 4 (A path with one critical level). *In this example, the existence of a critical level at $i = 4$ means that there is some choice in how to construct a tight geodesic path connecting the endpoints shown. The multicurve γ_{i+1} might be constructed from γ_i either by Dehn twisting around t_1 and t_2, or by Dehn twisting around t_3 and t_4. Note that γ_{i+1}*
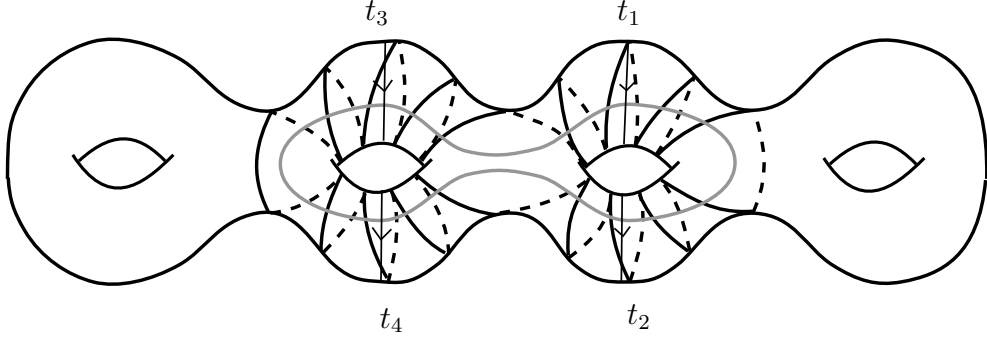
FIGURE 6. The endpoints of a path with one critical level.

can not be obtained by Dehn twisting around all four of t_1, t_2, t_3 and t_4, because then the edge corresponding to $\gamma_{i+1} - \gamma_i$ would not represent the boundary of an embedded subsurface of S.

It this last example, the critical level can be seen to partition the surface into two subsurfaces such that the projections to these two subsurfaces behave almost independently of each other. This observation is used later in section 3.

Also, unlike in these two simple examples, the number of critical levels will usually depend on the path, not just the endpoints.

The diameter of the space of tight geodesics can become unbounded. Example 4 can be used to show that the diameter of the space of geodesics can grow linearly with the distance between the vertices. For example, suppose γ_{2n+1} is constructed by Dehn-twisting γ_1 around each of t_1, t_2, t_3 and t_4 n times. There is a tight geodesic path, g_1, from c_2 to c_1 constructed by first untwisting the Dehn twists around t_1 and t_2, and then untwisting the Dehn twists around t_3 and t_4. Suppose g_2 is the tight geodesic path connecting γ_1 and γ_{2n+1} obtained by first untwisting the Dehn twists around t_3 and t_4, and then untwisting the Dehn twists around t_1 and t_4. The Hausdorff distance between g_1 and g_2 is as large as it could possibly be, namely $2n$, the length of the paths.

A uniform bound on the number of critical levels will now be obtained.

Lemma 5. *The number of critical levels is bounded from above by* $-6\chi(S) - 2$.

Proof. This lemma is proven by systematically checking through all the ways a critical level might occur. Each critical level is associated with at least one nontrivial component of $S\smallsetminus(\gamma_1 \cup \gamma_n)$, in such a way that the nontrivial components of $S\smallsetminus(\gamma_1 \cup \gamma_n)$ become used up.

Assume the representatives of the homotopy classes have been chosen as in subsection 2.1.

If i is a critical level, one of the following must occur
  (1) P_{i+1} has a component disjoint from P_i
  (2) There is a nontrivial component N of $S\smallsetminus(\gamma_n\cup\gamma_i)$ on the boundary of P_i such that ∂N and ∂P_i share an arc of γ_i.
  (3) P_{i+1} is obtained from P_i by attaching one or more handles.

Whenever (3) happens, either two or more components of P_i are contained in one component of P_{i+1}, or a component of P_i becomes an annulus or is changed by gluing one or more pairs of pants along the boundary. By construction, the boundary curves of the pants or annulus are not contractible in S.

In the worst case scenario, P_1 has one component, so it is not possible for a component of P_{i+1} to contain two or more components of P_i more times than (1) happens. Whenever (1) happens, the number of nontrivial components of $S\smallsetminus(\gamma_1 \cup \gamma_n)$ on the boundary of P_{i+1} is at least one more than the number of nontrivial components of $S\smallsetminus(\gamma_1 \cup \gamma_n)$ on the boundary of P_i. The largest possible number of nontrivial components of $S\smallsetminus(\gamma_1 \cup \gamma_n)$ is $-2\chi(S)$, from which the bound follows.

$\square$

3. Sublevel Projections

Subsurface projections were defined in [9] in order to be able to break the curve complex down into simpler pieces, thought of as curve or arc complexes of subsurfaces. The nested structure arising from the subsurface projections were used to describe families of quasigeodesics called hierarchy paths, and to show how these families of quasigeodesics are controlled by the subsurface projections of their endpoints.

In this section, the notion of sublevel projections are defined, so-named because there are some very strong parallels with subsurface

projections. Sublevel projections describe the degrees of freedom inherent in constructing geodesic paths in $\mathcal{C}(S, \alpha)$. Informally, given two multicurves m_1 and m_2 representing vertices in $\mathcal{C}(S, \alpha)$, their sublevel projections are a pair of homologous multicurves as close as possible to m_1 and m_2, whose subsurface projections to a subsurface of S defined by level sets of the overlap function coincide with the subsurface projections of m_1 and m_2.

In section 2 an algorithm for constructing middle paths was briefly outlined. Although, as discussed, these paths are not necessary paths in $\mathcal{C}(S, \alpha)$, the symmetry properties make these paths useful for defining subsurface projections. As discussed previously, in order to be able to meaningfully compare subsurfaces on which the overlap function takes on certain values, given the endpoints γ_1 and γ_n of a path, it is necessary to fix a canonical choice of representatives of the homotopy classes of the multicurves representing the path joining the endpoints. This is done as follows: In [7], lemma 8, it was shown that a middle path $\gamma_1, \ldots, \gamma_n$ has the property that for each $1 < i < n$, γ_i is an oriented, embedded subcomplex of the one dimensional cell complex $\gamma_1 - \gamma_n$, and the subsurface bounded by $\gamma_{i+1} - \gamma_i$ is $S_{n-i \leq f} \cup_j Int(c_j)$, where $S_{n-i \leq f}$ is the subsurface of S on which the overlap function, $f(\gamma_1, \gamma_n)$ is at least $n - i$, and $\{c_j\}$ are the contractible curves discarded from γ'_{i+1} after the surgery.

Given two integers $l_1 < l_2$ in the range of the overlap function of γ_1 and γ_n, the *sublevel projection* of γ_1 and γ_n between the levels l_1 and l_2, $\Pi_{l_1}^{l_2}(\gamma_1, \gamma_n)$, is the pair of homologous multicurves $(\gamma_{l_1+1}, \gamma_{l_2})$.

The sublevel projection of γ_1 and γ_n between the levels l_1 and l_2 is similar to a subspace projection to $S_{l_1+1 \leq f \leq l_2}$, in the sense that γ_{l_1} and γ_{l_2} represent vertices as close as possible to γ_1 and γ_n, respectively, given that they only intersect within the subsurface $S_{l_1+1 \leq f \leq l_2}$. It follows from theorem 9 in [7] that this definition is symmetric in γ_1 and γ_n.

The number of critical levels is not additive over subpaths.
In order to be able to compare different overlap functions, it was necessary to make a choice about representative of homotopy classes of multicurves. Unfortunately, this choice precludes the possibility of putting all the multicurves $\{\gamma_i\}$ in general and minimal position. It was also necessary to assume that the endpoints of the path were in general and minimal position, because otherwise the overlap function

is not a meaningful concept. For this reason, the number and type of
critical levels are not additive over subpaths.

Properties of Sublevel Projections. Consider the finite num-
ber of sublevel projections of the form $\Pi_i := \Pi_{l_i}^{l_{i+1}}(\gamma_1, \gamma_n)$, where l_i
and l_{i+1} are consecutive critical levels. It will be seen in the next sec-
tion that there is almost no choice in how to construct tight paths
between $\gamma_{l_{i+1}}$ and $\gamma_{l_{i+1}}$; the number of degrees of freedom for con-
structing tight geodesics is at a minimum. Any collection of surgeries
performed on the multicurve γ_i to construct a multicurve γ_{i+1} with
$d(\gamma_{i+1}, \gamma_n) = d(\gamma_i, \gamma_n) - 1$ necessarily decreases the distance between γ_i
and γ_n in one of the sublevel projections Π_i. A distance formula anal-
ogous to the distance formula from [9], with a uniform bound on the
number of sublevel projections follows immediately from the construc-
tion and Corollary 2. In this way, families of tight paths in $\mathcal{HS}(S, \alpha)$
are even more rigidly controlled by the sublevel projections of their
endpoints than is the case in the marking graph for hierarchy paths
under subsurface projections, [9].

4. Tangents to the space of tight geodesics

We now return to the proof of theorem 1.

Proof. Recall that all paths in $\mathcal{C}(S, \alpha)$ can be constructed by surger-
ing along horizontal arcs and discarding contractible curves and null
homologous multicurves. Also recall that, from the definition of tight,
it follows that for each i, the horizontal arcs are homotopic to arcs of
$\gamma_n \cap (S \smallsetminus \gamma_i)$. Without loss of generality, it can and will be assumed
that no surgeries are performed on a curve c along a horizontal arc a
with the property that a is homotopic to a subarc of c.

It was shown in the proof of theorem 5 of [7] that surgering γ_i along
a horizontal arc a and then surgering the resulting multicurve along a
horizontal arc $v_1 \circ a \circ v_2$, for vertical arcs v_1 and v_2 of $\gamma_n \cap (S \smallsetminus \gamma_i)$, nec-
essarily results in a multicurve that intersects γ_i, unless the arc a has
both endpoints on a curve c homotopic to the curve on which $v_1 \circ a \circ v_2$
has both its endpoints. If γ_i contains two or more curves $c_1, c_2 \ldots$
in the homotopy class $[c]$, and a has both endpoints on c_1, surgering
c_1 along a, and then c_2 along $v_1 \circ a \circ v_2$ does not define an edge in $\mathcal{C}(S, \alpha)$.

It seems to be general knowledge that Euler characteristic argu-
ments can be used to bound the number of homotopy classes of arcs

of $m_1 \cap (S \smallsetminus m_2)$, for two multicurves m_1 and m_2, where m_2 does not contain homotopic curves. For example, in [7], lemma 11 a bound of $-3\chi(S)$ is shown. In the case that γ_i does not contain homotopic curves, it follows that there is a bound on the number of horizontal arcs along which γ_i can be surgered to obtain γ_{i+1}.

If γ_i does contain homotopic curves, since it does not contain null homologous submulticurves, it must be the case that all freely homotopic curves γ_i have the same orientation. Let δ_i be the multicurve obtained by gathering all curves in γ_i in each homotopy class into a single curve. Lemma 11 of [7] applied to γ_n and δ_i gives a bound on the number of homotopy classes of horizontal arcs of $\gamma_n \cap (S \smallsetminus \gamma_i)$.

So which surgeries are optional and which surgeries are necessary to obtain a geodesic? A path $\{\gamma_1, \ldots, \gamma_n\}$ is a geodesic iff the maximum of the overlap function of γ_{i+1} with γ_n is one less than the maximum of the overlap function of γ_i with γ_n, for all i. *Necessary* surgeries are those that decrease the maximum of the overlap function, all other surgeries are *optional*. In example 4, a necessary surgery is either the surgery corresponding to S_{max} or the surgery corresponding to S_{min}. The existence of critical levels is a necessary condition for the existence of horizontal arcs not homotopic to arcs on the boundary of S_{max} or S_{min}, of which it has been shown that there can only be finitely many.

If γ_{l_i} and $\gamma_{l_{i+1}}$ are sublevel projections to consecutive critical levels, for each $l_i \leq j < l_{i+1}$ the only choice in how to construct tight geodesic paths is whether γ_{j+1} is obtained from γ_j by performing the surgery corresponding to the maximum or the minimum of the overlap function. These surgeries do not always construct homotopic multicurves as in example 3.

The upper bound on the dimension of the space of tight geodesic paths connecting γ_1 and γ_n was obtained by assuming the worst case scenario; namely, that all but the two homotopy classes, one on ∂S_{max} and the other on ∂S_{min}, of the maximal number of horizontal arcs of $\gamma_n \cap (S \smallsetminus \gamma_i)$ give rise to optional surgeries. This bound is clearly not sharp.

$\square$

Jacobifields. Given a path $\{\gamma_1, \ldots, \gamma_n\}$ and a surgery performed on γ_i, a discrete version of an element in the tangent space to the space

of geodesics can be defined. The basic problem is how to define what
is meant by "the same" surgery when it is performed on different mul-
ticurves. This is used to parameterise the space of geodesics.

Jacobi fields are vector fields defined along families of geodesics.
When restricted to a geodesic, a Jacobi field defines an infinitesimal
deformation of the geodesic within a one-parameter family of geodesics.
On a curve complex, a Jacobi field J is a map defined on a familiy of
tight geodesic paths. When restricted to a tight geodesic path γ^0, it
is a map from the vertices of the path, $\gamma_1^0, \ldots, \gamma_n^0$, onto edges, where
$J(\gamma_i^0)$ is the edge joining γ_i^0 to γ_i^1, for a tight geodesic path $\gamma_1^1, \ldots, \gamma_n^1$.
The sum of two Jacobi fields J_1 and J_2 is a Jacobi field if for each γ_i,
the edges $J_1(\gamma_i)$ and $J_2(\gamma_i)$ represent disjoint subsurfaces of S, S_1 and
S_2, such that the boundary of $S_1 \cup S_2$ is the union of two multicurves,
γ_i and δ_i.

Given a path $\{\gamma_1, \ldots, \gamma_n\}$ and a surgery performed on γ_i, a discrete
version of an element in the tangent space to the space of geodesics can
be defined. The basic problem is how to define what is meant by "the
same" surgery when it is performed on different multicurves. This is
used to parameterise the space of geodesics.

Assume the choice of representatives of homotopy classes of the
multicurves $\{\gamma_1, \ldots, \gamma_n\}$ defined in subsection 2.1. Let $a_1, a_2, \ldots$ and
$h_1, h_2, \ldots$ be horizontal arcs of $\gamma_n \cap (S \smallsetminus \gamma_i)$. In subsection 2.1 it was
shown that the subspace of S inside which γ_i is altered by the surgeries
$a_1, a_2 \ldots$ to obtain γ_{i+1} is an ϵ-neighbourhod of $\cup_m R_{a_m} \cup_l (G_l \cup R_{b_l}) \cup_k$
$d_k \cup_j n_j$. So whenever $\cup_m R_{a_m} \cup_l (G_l \cup R_{b_l}) \cup_k d_k \cup_j n_j$ is disjoint from the
subspace of S in which γ_i is altered by the surgeries along $h_1, h_2, \ldots$, the
horizontal arcs $h_1, h_2, \ldots$ of $\gamma_n \cap (S \smallsetminus \gamma_i)$ also define surgeries along γ_{i+1}.

Let $\{\gamma_1, \ldots, \gamma_n\}$ be a tight geodesic path. Basically, this is con-
structed by performing one set of necessary surgeries and a selection of
optional surgeries on γ_i to obtain γ_{i+1}, for all $1 \leq i < n$. Suppose h_1
is a horizontal arc representing an optional surgery on γ_i, then a cor-
responding optional surgery can be defined on all γ_j with the property
that P_j is disjoint from R_{h_1}. Let γ_j' be the multicurve γ_j surgered along
h_1, with any resulting contractible or null homologous submulticurves
discarded. So suppose the surgery can be defined along a subinterval
$[s, e]$ of the path, a Jacobi field $J(h_1, g_1)$ is defined to be tangent to the
edge $\gamma_j' - \gamma_j$ for j in a subinterval of $[s, e]$ of the form $[g_1, e]$, and zero

elsewhere.

It is possible to define the inverse of a surgery. However, performing a surgery and then later performing its inverse would result in a path that is not tight. In order to obtain a tight geodesic, once a surgery has been performed, it can not be undone, hence the choice of the subinterval on which the Jacobi field has its support.

A family of tight geodesic paths connecting γ_1 and γ_n is obtained as follows. When $g \leq i < e$ is not a critical level along the path $\gamma_1, \ldots, \gamma_{g-1}, \gamma_g', \ldots, \gamma_e', \gamma_{e+1}, \ldots, \gamma_n$, there is a horizontal arc h_2 of $\gamma_n \cap (S \smallsetminus \gamma_i')$, where h_2 is a concatenation $v_1 \circ h_1 \circ v_2$ of arcs of $\gamma_n \cap (S \smallsetminus \gamma_i)$. Then h_2 defines a Jacobi field $J(h_2, g_2)$ along $\gamma_1, \ldots, \gamma_{g-1}, \gamma_g', \ldots, \gamma_e', \gamma_{e+1}, \ldots, \gamma_n$. Repeating this results in a one parameter family of tight geodesic paths connecting γ_1 and γ_n.

Combining Jacobi Fields. Not all sums of Jacobi fields are defined whenever the associated subspaces of S are disjoint. Let $\{a_1, \ldots, a_k\}$ be horizontal arcs of $\gamma_n \cap (S \smallsetminus c)$ for a curve c. Ignoring contractible curves, it is not hard to show that the curve c, when surgered along the arcs $\{a_1, \ldots, a_k\}$ has zero geometric intersection number with c iff all the arcs are either to the left of c or to the right of c. It follows that the sum of two Jacobi fields will be a Jacobi field iff, for each vertex of the paths along which the Jacobi fields are nonzero, the horizontal arcs representing the surgeries are all on the same side of each curve in the multicurve representing the vertex.

5. Cusps and Horospheres

Instead of studying a complex whose vertices are represented by homologous multicurves, an alternative complex could have been defined, in which the vertices are represented by homologous curves, as in [1] for example. However, algorithms for constructing paths in such complexes are obtained from surgeries on curves that construct multicurves, and not curves. Even though multicurves are not allowed to contain null homologous submulticurves, and α is assumed to be primitive, when the genus g of any connected component of S is at least 3, a path in $\mathcal{C}(S, \alpha)$ can pass through vertices representing multicurves with arbitrarily many connected components. An example of this is [7], figure 12. Informally, this happens when a path becomes stuck in a subsurface of S, which will be used to obtain subcomplexes closed under the

action of stabiliser subgroups of the Torelli group.

Cusps and Horoballs. The *cusp*, $\mathcal{CP}_g(c, \alpha)$, corresponding to the homotopy class of the oriented curve c, is the largest subcomplex of $\mathcal{C}(S, \alpha)$ whose vertices all correspond to multicurves with more than one curve homotopic to c. Note that in genus 2 there are no cusps.

Let $\hat{i}(a, b)$ denote the algebraic intersection number of the curves a and b.

Theorem 6. *For $g \geq 3$, $\mathcal{C}(S, \alpha)$ contains a cusp $\mathcal{CP}_g(c, \alpha)$ whenever $\hat{i}(c, \alpha) = 0$, $[c] \neq 0$ and $\pm[c] \neq \alpha$.*

So by choosing α accordingly, a subcomplex closed under the action of a given stabiliser subgroup can always be found.

Proof. This is shown by constructing a vertex in the cusp, from which it follows that the cusp can not be empty. Let a be a curve in $[\alpha]$ disjoint from c, whose existence is ensured by the assumptions on c. Since a can not be given an orientation to make it homologous to c, there is a horizontal arc in S, with one endpoint on a curve $-c$ and the other endpoint on the curve a, and whose interior is disjoint from the curves a and c. Let a_1 be the curve obtained by surgering a and $-c$ along this horizontal arc. Similarly, there is a horizontal arc with endpoints on a_1 and $-c$, with interior disjoint from a_1 and $-c$. Let a_2 be the curve obtained by surgering a_1 and $-c$ along this horizontal arc, etc. Then $a_n \cup nc$ is a multicurve representing a vertex in $\mathcal{CP}_g(c, \alpha)$. $\square$

Theorem 7. *The complex $\mathcal{C}(S, \alpha)$ with the cusps cut off is connected and contractible.*

Proof. Connectivity is a corollary of theorem 1.9 of [10] for genus at least 3, and contractibility follows from the argument in [6]. For genus 2 there are no cusps, so connectivity and contractibility was established in [2]. $\square$

The *horoball*, $\mathcal{HB}_g(nc, \alpha)$, is the largest subcomplex of the cusp $\mathcal{CP}_g(c, \alpha)$ whose vertices all represent multicurves with exactly n curves homotopic to c, where $n > 1$.

Theorem 8. *Horoballs are connected and contractible embedded sub-complexes of $\mathcal{C}(S, \alpha)$. The horoballs $\mathcal{HB}_g(nc, \alpha)$, $\mathcal{HB}_g(n_1 c_1, \alpha), \ldots, \mathcal{HB}_g(n_k c_k, \alpha)$*

intersect whenever all the c_i are disjoint, $S \smallsetminus (c_1, \ldots, c_k)$ is connected and α is not contained in the span of $\{[c], [c_1], \ldots, [c_k]\}$. The intersection of horoballs is connected and contractible.

Proof. Connectivity and contractibility of the horoballs and their intersections can be shown similiarly as for cusps, and when $S \smallsetminus (c_1, \ldots, c_k)$ is connected, the existence of intersections of horoballs is proven by the same argument used to show the existence of cusps.

$\square$

Let $\mathcal{T}(c)$ be the subgroup of the Torelli group of S that stabilises the curve c, and, similarly, let $\mathcal{T}(c, \ldots, c_k)$ be the subgroup of the Torelli group that stabilises the curves $\{c, \ldots, c_k\}$. The horoball $\mathcal{HB}_g(nc, \alpha)$ is a subcomplex of $\mathcal{BC}(S, \alpha)$ closed under the action of $\mathcal{T}(c)$. The intersection of the horoballs $\mathcal{HB}_g(nc, \alpha), \ldots, \mathcal{HB}_g(n_k c_k, \alpha)$ is denoted by $\mathcal{HB}_g(nc, \ldots, n_k c_k, \alpha)$. The subcomplex $\mathcal{HB}_g(nc, \ldots, n_k c_k, \alpha)$ is closed under the action of $\mathcal{T}(c, \ldots, c_k)$.

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